Sheet 10

(1) Assume $\exists\, \alpha, \beta : \mathbb{R} \to \mathbb{R}$ $(\beta \geq 0, \beta(0) = 0)$ such that <u>for all</u> $\varphi \in C^\infty(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}))$, the function

$$u(t,x) = \alpha(r)\,\varphi(t - \beta(r)), \quad (r = |x|)$$

satisfies the wave equation $u_{tt}(t,x) - \Delta u(t,x) = 0 \quad \forall (t,x) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$

<u>Show:</u> This is only possible if $n = 1$ or $n = 3$ (and determine α, β).

$$u_{tt}(t,x) = \alpha(r)\varphi''(t - \beta(r))$$

Recall (Sheet 3) that if $v(x) = f(|x|)$, then

$$\Delta v(x) = f''(r) + \frac{n-1}{r} f'(r)$$

Hence, letting $f(r) = \alpha(r)\varphi(t - \beta(r))$ (fixed t), we have

$$f'(r) = \alpha'(r)\varphi(t-\beta(r)) - \alpha(r)\beta'(r)\varphi'(t-\beta(r))$$

$$f''(r) = \alpha''(r)\varphi(t-\beta(r)) - 2\alpha'(r)\beta'(r)\varphi'(t-\beta(r)) + \alpha(r)(\beta'(r))^2\varphi''(t-\beta(r))$$

So

$$\Delta u(t,x) = \alpha''(r)\varphi(t-\beta(r)) - 2\alpha'(r)\beta'(r)\varphi'(t-\beta(r)) + \alpha(r)\beta'(r)^2\varphi''(t-\beta(r)) + \frac{n-1}{r}\left(\alpha'(r)\varphi(t-\beta(r)) - \alpha(r)\beta'(r)\varphi'(t-\beta(r))\right)$$

$\overset{u \text{ satisfies}}{\underset{\text{wave equation}}{=}} u_{tt}(t,x) = \alpha(r)\varphi''(t-\beta(r))$

Rearrange (collecting $\varphi, \varphi', \varphi''$ terms):

$$(1)\begin{cases} \left(\alpha''(r) + \frac{n-1}{r}\alpha'(r)\right)\varphi(t-\beta(r)) \\ + \left(-2\alpha'(r)\beta'(r) - \frac{n-1}{r}\alpha(r)\beta'(r)\right)\varphi'(t-\beta(r)) \\ + \left(\alpha(r)(\beta'(r))^2 - \alpha(r)\right)\varphi''(t-\beta(r)) = 0 \end{cases}$$

This has to hold for all $\varphi \in C^\infty$. Take φ to be a polynomial, so that $\varphi, \varphi', \varphi''$ are linearly independent.

$\Rightarrow$ This means coefficients of $\varphi, \varphi', \varphi''$ in (1) must be identically 0.

i.e. (I) $\alpha''(r) + \frac{n-1}{r}\alpha'(r) = 0$

(II) $\left(2\alpha'(r)\beta'(r) + \frac{n-1}{r}\alpha(r)\right)\beta'(r) = 0$

(III) $\alpha(r)\left(\beta'(r)^2 - 1\right) = 0$ $\left(\forall r \in [0,\infty)\right)$

- From (I), we see α must have general solution $\alpha(r) = K r^{\gamma}$ $n \geq 3$ some $K, \gamma \in \mathbb{R}$. (See sheet 3) (or $\alpha(r) = a \log r$ for $n=2$ or α lines $\pm c_2$ for $n=1$.)
- Hence From (III) we can divide by α to see $(\beta'(r))^2 - 1 = 0$

So $\beta'(r) = \pm 1$.

Assume β C^1: we have $\beta(r) = c \pm r$.

But $\beta(0)=0$, $\beta \geq 0$. So $\underline{\beta(r) = r}$ (hence $\beta'(r) = 1$, β''

~~Thus (I) and (II) become: (using $\alpha(r) = Kr^{\gamma}$)~~

~~$\begin{cases} K\gamma(\gamma-1)r^{\gamma-2} + (n-1)K\gamma r^{\gamma-2} = 0 \\ 2K\gamma r^{\gamma-1} + \end{cases}$~~

(II) becomes $2\alpha'(r) + \frac{n-1}{r}\alpha(r) = 0$

This has general solution $\alpha(r) = K r^{\gamma}$

need: i.e. $2\gamma K r^{\gamma-1} + (n-1)K r^{\gamma-1} = 0$ so $2\gamma + (n-1) = 0$

So $\gamma = -\frac{(n-1)}{2}$

We also need this α to satisfy (I). i.e., need

$K\gamma(\gamma-1)r^{\gamma-2} + K(n-1)\gamma r^{\gamma-2} = 0$

ie $\gamma(\gamma-1) + (n-1)\gamma = 0$

Plug in γ: $-\frac{(n-1)}{2}\left(-\frac{(n-1)}{2} - 1\right) + -\frac{(n-1)}{2}^2 = 0$

ie $\frac{(n-1)}{2}\frac{(n+1)}{2} - \frac{1}{2}(n-1)^2 = 0$

$\times 4$: $n^2 - 1 - 2(n^2 - 2n + 1) = 0$

$n^2 - 4n + 3 = 0$

$(n-1)(n-3) = 0$. ~~This~~ Thus we need $\underline{n=1 \text{ or } n=3}$ and

$\beta(r) = r$

$\alpha(r) = \begin{cases} K & n=1 \\ Kr^{-1} & n=3 \end{cases}$, $K \in \mathbb{R}$

② $g = \begin{pmatrix} -1 & 0 \\ 0 & 1_{1_1} \end{pmatrix} \in \mathbb{R}^{4\times 4}$ (note $g = g^t = g^{-1}$)

$\Lambda \in \mathbb{R}^{4\times 4}$ is a Lorentz transformation iff $\Lambda^T g \Lambda = g$.

(a) If Λ_1, Λ_2 are Lorentz transformations, then

$$(\Lambda_1 \Lambda_2)^T g (\Lambda_1 \Lambda_2) = \Lambda_2^T (\Lambda_1^T g \Lambda_1) \Lambda_2$$

(matrix mult is associative!) $= \Lambda_2^T g \Lambda_2$

$= g.$

So $\Lambda_1 \Lambda_2$ is a ~~linear~~ Lorentz transformation

(b) From linear algebra: $A \in \mathbb{R}^{4\times 4}$ is invertible iff $\text{rank}(A) = 4$.

If $A, B \in \mathbb{R}^{4\times 4}$, $\text{rank}(AB) \leq \text{rank}(A), \text{rank}(B)$

If Λ is a Lor.T., then $\text{rank}(\Lambda^T g \Lambda) = \text{rank}(g) = 4$.

Hence $\text{rank}(\Lambda) = 4$, so Λ is invertible. (Λ^{-1} exists)

Now note that

$$(\Lambda^{-1})^T g \Lambda^{-1} = (\Lambda^{-1})^T \Lambda^T g \Lambda \Lambda^{-1}$$

$$= (\Lambda^T)^{-1} \Lambda^T g \Lambda \Lambda^{-1}$$

$= I g I = g.$ So Λ^{-1} is a Lorentz trans.

(c) For $x, y \in \mathbb{R}^4$, define $\langle x, y \rangle_g := x^T g y$ $(= x \cdot (gy) \in \mathbb{R})$

Let Λ be a Lorentz transformation. Then

$$\langle \Lambda x, \Lambda y \rangle_g = (\Lambda x)^T g (\Lambda y)$$

$$= x^T \Lambda^T g \Lambda y$$

$$= x^T g y = \langle x, y \rangle_g.$$

(d) (i) Q orthogonal : $QQ^T = Q^T Q = I$.

$$\Lambda = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & Q & \\ 0 & & & \end{array}\right)$$

$$\Lambda^T g \Lambda = \left(\begin{array}{c|c} 1 & \\ \hline & Q^T \end{array}\right) g \left(\begin{array}{c|c} -1 & \\ \hline & I \end{array}\right) \left(\begin{array}{c|c} 1 & \\ \hline & Q \end{array}\right) = \left(\begin{array}{c|c} 1 & \\ \hline & Q \end{array}\right) \left(\begin{array}{c|c} -1 & \\ \hline & Q \end{array}\right)$$

$$= \begin{pmatrix} -1 & \\ & Q^T Q \end{pmatrix} = \begin{pmatrix} -1 & \\ & I \end{pmatrix} = g.$$

(ii) $(t,x) \to (-t,x)$ has matrix g!

$$g^T g g = g^{-1} g g = g.$$

(iii) Write $\gamma = \frac{1}{\sqrt{1-a^2}}$. Then $\Lambda : (t,x) \to (\gamma(t - a x_1), \gamma(x_1 - at), x_2, x_3)$

So $\Lambda = \begin{pmatrix} \gamma & -a\gamma & 0 & 0 \\ -a\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (considering where Λ maps e_0, e_1, e_2, e_3)

$= \Lambda^T$.

Check $\Lambda^T g \Lambda$ maps each e_i to e_i for $1 \le i \le 3$ (x variables)

and e_0 to $-e_0$ (t variable).

$e_0 \overset{\Lambda}{\longmapsto} (\gamma, -a\gamma, 0, 0) \overset{g}{\longrightarrow} (-\gamma, -a\gamma, 0, 0)$

$e_0 = (1,0,0,0)$

$\overset{\Lambda^T}{\longrightarrow} (-\gamma^2 + a^2\gamma^2, a\gamma^2 - a\gamma^2, 0, 0)$

$= (a^2 - 1)\gamma^2 e_0 = -e_0$

$e_1 \overset{\Lambda}{\longrightarrow} (-a\gamma, \gamma, 0, 0) \overset{g}{\longrightarrow} (a\gamma, \gamma, 0, 0) \overset{\Lambda^T}{\longrightarrow} (a\gamma^2 - a\gamma^2, -a^2\gamma^2 + \gamma^2, 0, 0)$

$= (1 - a^2)\gamma^2 e_1 = e_1$

$e_2 \overset{\Lambda^T g \Lambda}{\longrightarrow} e_2$ (easy)

$e_3 \overset{\Lambda^T g \Lambda}{\longrightarrow} e_3$.

So $\Lambda^T g \Lambda = g$.

(e) Suppose $u \in C^2(\mathbb{R} \times \mathbb{R}^3)$ solves $u_{tt}(t,x) - \Delta u(t,x) = 0 \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^3$

Then $\operatorname{trace}(g D^2 u(t,x)) = 0$

$$\left(D^2 u = D^2_{(t,x)} u = \begin{pmatrix} u_{tt} & u_{tx_1} & \cdots & u_{tx_3} \\ \vdots & & & \\ u_{x_3 t} & \cdots & \cdots & u_{x_3 x_3} \end{pmatrix} \right)$$

Let Λ be a Lorentz transformation and define

$$v(t,x) := u(\Lambda(t,x))$$

Now note $D^2 v(t,x) = \Lambda^T D^2 u(\Lambda(t,x)) \Lambda$ (chain rule, linear algebra)

$-v_{tt}(t,x) + \Delta v(t,x)$

So $\overset{''}{tr}(g D^2 v(t,x)) = tr(g \Lambda^T D^2 u(\Lambda(t,x)) \Lambda)$

trace invariant under cyclic permutations $= tr(\Lambda g \Lambda^T D^2 u(\Lambda(t,x)))$

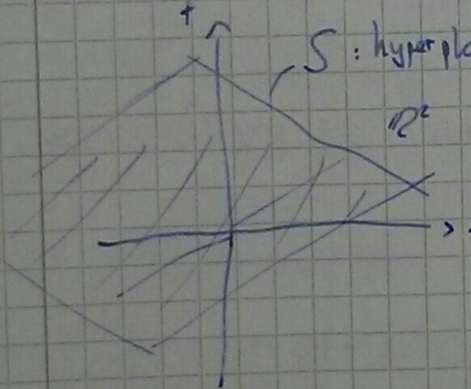
$= tr(g D^2 u(\Lambda(t,x)))$

$= -u_{tt}(\Lambda(t,x)) + \Delta_x u(\Lambda(t,x))$

$= 0$

So v satisfies wave equation on $\mathbb{R}\times\mathbb{R}^3$ too.

(3) Let $S = \{(t,x) \in \mathbb{R}^4 : t = ax_1\}$ $0<a<1$



S: hyperplane in $\mathbb{R}^4$. $\varphi(x) = ax_1$, $1-|\nabla\varphi|^2 = 1-a^2>0$

S is spacelike

Show that finding $u: \mathbb{R}^4 \to \mathbb{R}$ s.t.

$$(2) \begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}^4 \\ u = g,\ u_t = h & \text{on } S \end{cases}$$

is equivalent to solving a (ie maybe reduced to) wave equation with initial data on $\{0\}\times\mathbb{R}^3$.

Let Λ be the Lorentz boost from Q.2 (d) (iii).

Let $v = u \circ \Lambda$

Then if u ~~solves (2)~~, satisfies $u_{tt} - \Delta u = 0$ on $\mathbb{R}^4$,

so does v (by Q2 (e)).

Now note that if $(t,x) \in S$, then
$(t,x) = (ax_1, x_1, x_2, x_3)$

So $\Lambda(t,x) = \left(0, \frac{x_1 - a^2 x_1}{\sqrt{1-a^2}}, x_2, x_3\right)$

$= \left(0, \sqrt{1-a^2}\, x_1, x_2, x_3\right) \in \{0\} \times \mathbb{R}^3$

So ~~u~~ v solves (3) $\begin{cases} v_{tt} - \Delta v = 0 & \text{on } \mathbb{R}^4 \\ v = \tilde{g},\ v_t = \tilde{h} & \text{on } \{0\}\times\mathbb{R}^3 \end{cases}$

~~$t \in S$,~~ ~~$\tilde{g}(x) =$~~ ~~$g(x)$~~ $\tilde{g}(x) = g(\underbrace{\Lambda^{-1}(0,x)}_{\in S})$

~~$g(t,x) = g(\Lambda^{-1}\Lambda$~~

Conversely, if v solves (3), then define $u(t,x) = v(\Lambda(t,x))$
solves (2).

$\left(u(t,x) = u(\Lambda^{-1}\Lambda(t,x)) = v(\Lambda(t,x)) \right)$ //